Conditions for Mean and First-Moment Stability of Positive Markov Jump Linear Systems with Time-Varying Subsystems

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***Abstract*— This work studies mean stability and first-moment stability of discrete-time positive Markov Jump Linear Systems with time-varying discrete modes. We adopt an approach based on linear co-positive Lyapunov functions that produces two sets of non-equivalent sufficient conditions with guaranteed exponential decay rates. Due to the general time-varying nature of the subsystems, the conditions require infinitely many tests. Hence, we show how one of the two introduced conditions can be finitely tested in the special case where the subsystems take uncertain values within polytopes.**

***Index Terms*— Markov Jump Linear Systems, Switching Systems, Time-varying systems, Stability of Linear Systems.**

I. Introduction

Markov Jump Linear Systems (MJLSs) are described by switched models whose switching rule follows a Markovian stochastic process. This class of linear systems has been largely studied in the last decades due to some remarkable applications to various problems arising in engineering, economics, biology, etc. [1], [2], [3]. A quite established theory exists to date (see, e.g., the monograph [2]) encompassing some of the most popular problems in analysis and control.

Positive MJLSs (PMJLSs) are a popular subclass in which the state variables can only be non-negative, provided that non-negative inputs and initial state are applied to the system. PMJLSs have been popular due to the inherent positivity of many systems of practical interest, as for example population models, consensus models with switching topology, etc. [3]. Analysis and control of PMJLSs are themselves mature fields with a rich and insightful literature (see, e.g., the recent [4], [5], [6], [7], [8] and references therein).

Due to their stochastic nature, MJLSs allow for various characterizations of stability. To name a few, literature has studied stochastic stability, p-moment stability, almost-sure stability, etc. [2]. Mean stability, which is quite easy to characterize and test for MJLSs with time-homogeneous transition rates, has no particular meaning for general MJLSs, as the convergence to zero of the expected value of the state variable does not yield practically meaningful properties on its evolution. On the other hand, when positivity is imposed on the system's dynamics, mean stability is equivalent to first-moment stability and implies that almost all the realizations asymptotically converge to zero, that is, almost sure stability, a significant practical property [3].

The core idea of this paper, and the main motivation behind it, is to provide stability conditions for PMJLSs with time-varying discrete modes, which have been less addressed in the literature. Albeit being studied for control purposes [2], literature on stability results for the general time-varying case is scarce (one can mention [9] for the continuous-time case with no positivity assumptions). More attention has been paid to the case of MJLSs with dynamic matrices subject to polytopic uncertainties (see, e.g., [10], [11]) or parameter-varying [12]. Considering time-varying subsystems in MJLSs allows for modeling complex systems in which a finite number of time-varying behaviors is selected according to a stochastic rule. Consider, for example, an epidemiological model with a finite number of concurring scenarios that exhibit a time-varying behavior (such as seasonality).

Our work starts by providing a general overview of the main stability definitions for MJLSs (and PMJLSs) and by discussing their relationships. Then, we examine how linear co-positive Lyapunov functions (LCLFs) can provide sufficient conditions of exponential mean stability (and equivalently exponential 1-moment stability) for PMJLSs with time-varying subsystems. For these conditions, we also provide guaranteed exponential convergence rates (on the mean value of the state and on its first moment, respectively). As expected, due to the generality of the considered class of systems, the conditions involve infinitely many tests in the general case. Hence, we study the special and finitely testable case in which the dynamic matrices take uncertain values within polytopes. Remarkably, we show that only one of the two non-equivalent LCLF-based conditions provided for the general time-varying case can be adapted to a finitely testable condition for the polytopic scenario. To the best of our knowledge, all these contributions are novel and appear here for the first time. Some of these can be seen as the time-varying extension of the results in [5] when considered for time-inhomogeneous transition probabilities.

Notation and preliminaries: We consider a probability space $(\Omega, \mathcal{F}, P)$ consisting of the sample space Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω, and a probability measure P on it. For any event $F \in \mathcal{F}$, $\mathbb{1}_F$ is the usual indicator function, defined as $\mathbb{1}_F(\omega) = 1$, if $\omega \in F$, and 0 otherwise.

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$\mathbb{N}_0$ is the set of natural numbers including 0. $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) is the set of non-negative (positive) real numbers. $\mathbb{R}^n_{\geq 0}$ ($\mathbb{R}^n_{>0}$) is the non-negative (positive) orthant of $\mathbb{R}^n$. $\mathbb{R}^{m\times n}_{\geq 0}$ is the cone of non-negative $m\times n$ matrices. I_n is the $n\times n$ identity matrix and $\mathbf{1}_n$ is the column vector consisting of n ones. Vectors are defined as columns, so that $v^\top$ denotes a row vector of suitable size. For sets of vectors defined with indexes the notation $[v_i]_h$ will be used to denote the h-th entry of vector v_i. Inequalities among vectors and matrices of the same dimensions have to be understood component-wise, i.e., $M \leq N$ ($M < N$) if $m_{ij} \leq n_{ij}$ ($m_{ij} < n_{ij}$) for all i, j, where m_{ij} denotes the i, j entry of matrix M. In this sense, a non-negative matrix $M \in \mathbb{R}^{m\times n}_{\geq 0}$ is also denoted by $M \geq 0$, where 0 is the matrix of appropriate dimensions whose entries are all zero. $\sigma(M)$ and $\rho(M)$ respectively denote the spectrum and the spectral radius of a square matrix M, which is said to be *Schur-stable* if $\sigma(M) \subset \{z \in \mathbb{C} : |z| < 1\}$ or, equivalently, if $\rho(M) < 1$. The following conditions are equivalent for the Schur-stability of a non-negative $M \in \mathbb{R}^{n\times n}_{\geq 0}$ [13]:

- $\rho(M) < 1, \quad (1)$
- $I - M$ is non-singular and $(I - M)^{-1} \geq 0, \quad (2)$
- $\exists\, v \in \mathbb{R}^n_{>0}$ s.t. $v^\top M < v^\top, \quad (3)$
- $\exists\, w \in \mathbb{R}^n_{>0}$ s.t. $Mw < w. \quad (4)$

The last two conditions can be tested via Linear Programming (LP), making them appealing from the computational viewpoint. Condition (3) is equivalent to the existence of a linear co-positive Lyapunov function (LCLF) $V(x(k)) = v^\top x(k)$ for the discrete-time positive difference equation $x(k+1) = Mx(k)$, with $x(0) \in \mathbb{R}^n_{\geq 0}$, whereas (4) asks the same on the system with one-step transition matrix $M^\top$.

The symbols $\|x\|$ and $\|M\|$ denote a generic norm for vector x and for the corresponding induced norm of matrix M. $\|x\|_1$ denotes the 1-norm of vector $x \in \mathbb{R}^n$, i.e., $\|x\|_1 = \sum_{i=1}^n |x_i|$. Clearly, if $x \in \mathbb{R}^n_{\geq 0}$, $\|x\|_1 = \sum_{i=1}^n x_i = \mathbf{1}_n^\top x$.

For two matrices $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{p\times q}$ the Kronecker product is denoted by $M \otimes N$, with:

$$M \otimes N = \begin{bmatrix} m_{11}N & \cdots & m_{1n}N \\ \vdots & \ddots & \vdots \\ m_{m1}N & \cdots & m_{mn}N \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

Finally, $\mathrm{diag}_i[M_i]$ will denote a block diagonal matrix obtained by orderly putting the $M_1, \ldots, M_p$ matrices on the diagonal blocks.

II. Problem formulation and Preliminary results

We deal with discrete-time MJLSs with time-varying subsystems, whose state evolution is described by:

$$\begin{aligned} x(k+1) &= A_{\theta(k)}(k)x(k), \quad k \geq 0, \\ x(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{5}$$

where $\{\theta(k),\, k \in \mathbb{N}_0\}$ is a discrete-time stochastic process described by a Markov chain taking values in the finite set $\underline{N} := \{1, 2, \ldots, N\}$, to which system matrices $\{A_1(k), \ldots, A_N(k)\}$ correspond. The probability of transition from mode i to mode j at times $k, k+1$ is given by:

$$P(\theta(k+1) = j|\theta(k) = i) = \pi_{ij}, \quad i, j \in \underline{N}, \tag{6}$$

and the probabilities $\pi_{ij} \geq 0$ form the transition probability (TP) matrix $\Pi = [\pi_{ij}]$, with $\sum_{j=1}^N \pi_{ij} = 1$, or equivalently $\Pi\, \mathbf{1}_N = \mathbf{1}_N$. The initial probability distribution on $\theta(k)$ is denoted $p_{\theta(0)} = [P(\theta(0) = 1) \ \cdots \ P(\theta(0) = N)]^\top$.

Definitions and conditions of positivity for system (5) are well-known in literature for the case of constant subsystems (see, e.g., [5]). The extension to MJLSs with time-varying subsystems is straightforward and follows.

Definition 1: A system described by (5) is a *positive* Markov Jump Linear System (PMJLS) if, for any initial condition $x_0 \geq 0$, it holds that $x(k) \geq 0$ for all $k > 0$.

A simple necessary and sufficient condition can be formulated on $\{A_1(k), \ldots, A_N(k)\}$ to satisfy this definition.

Proposition 1: System (5) is a PMJLS if and only if for all $i \in \underline{N}$, $A_i(k) \geq 0$ at all $k \geq 0$.

Various definitions of stability can be proposed for MJLSs, with remarkable relationships holding in case of positivity.

Let us start with the definition of almost sure stability [3], which ensures that nearly all the realizations asymptotically tend to zero, a property of clear, practical interest.

Definition 2: The MJLS described by (5) is *almost surely asymptotically stable* if, for any $x_0 \in \mathbb{R}^n$ and any initial probability distribution $p_{\theta(0)}$,

$$P\left(\lim_{k\to\infty} \|x(k)\| = 0\right) = 1, \tag{7}$$

and is *almost surely exponentially stable* if, for some $\rho > 0$,

$$P\left(\limsup_{k\to\infty} \frac{1}{k} \log \|x(k)\| \leq -\rho\right) = 1. \tag{8}$$

Another characterization that has been extensively studied for MJLSs is moment stability.

Definition 3: The MJLS described by (5) is *p-moment stable* if, for any $x_0 \in \mathbb{R}^n$ and any initial probability distribution $p_{\theta(0)}$:

$$\lim_{k\to\infty} E[\|x(k)\|^p] = 0. \tag{9}$$

Moreover, the *p-moment stability* is *exponential* if there also exist a positive scalar α, and $\beta \in (0, 1)$, such that $E[\|x(k)\|^p] \leq \alpha\beta^k \|x_0\|^p$ for all times $k \geq 0$.

p-moment stability has two important cases in $p = 1$ (first-moment stability, 1-moment stability in brief) and $p = 2$ (mean square stability, MSS). Moment stability of any order implies almost sure stability and higher moment stability implies lower moment stability [14]. Most of literature dealt with MSS, as necessary and sufficient condition to test it exist (at least for the special case of constant subsystems). This explains why MSS is often used to assess almost sure stability, which is more difficult to test, even if MSS is a way stronger condition. Clearly, this also motivates the interest in studying 1-moment stability, which still implies almost sure stability and is less stringent than MSS.

The concept of *mean stability*, that is the convergence to zero, on average, of the realizations of $x(k)$, has no particular meaning for general MJLSs. On the other hand, for *positive* MJLSs mean stability can be shown equivalent to 1-moment stability, and this means that it implies almost sure stability; these facts have been investigated and proved in various works, still for the case of constant subsystems (see, e.g., [3], [6]). The equivalence of mean and 1-moment stability for PMJLSs is appealing, since mean stability is relatively easy to characterize and test. The following definition holds under positivity constraints.

Definition 4: A PMJLS as in (5) is *mean stable* if, for any $x_0 \in \mathbb{R}^n_{\geq 0}$ and any initial probability distribution $p_{\theta(0)}$:

$$\lim_{k\to\infty} E[x(k)] = 0. \tag{10}$$

Moreover, the mean stability is *exponential* if there also exists a positive scalar α, and $\beta \in (0,1)$, such that $\|E[x(k)]\| \leq \alpha\beta^k\|x_0\|$ for all $k \geq 0$.

We start with a preliminary result on the usage of common LCLFs to investigate the exponential stability of a discrete-time positive system with time-varying matrices with no Markovian switching rule. The result has appeared before (it is, for example, a special case of the results for time-delay systems of [15]). A proof (different from the one in the cited reference) is reported here for the reader's convenience.

Proposition 2: Consider a time-varying positive system described by

$$\begin{aligned} x(k+1) &= A(k)x(k), \quad k \geq 0, \\ x(0) &= x_0 \in \mathbb{R}^n_{\geq 0}, \end{aligned} \tag{11}$$

with $A(k) \geq 0$ at all $k \geq 0$. The system is exponentially stable with a guaranteed decay rate $\beta \in (0,1)$ if either one among the following two (non-equivalent) conditions holds:

- $\exists v \in \mathbb{R}^n_{>0}, \exists \beta_v \in (0,1)$ s.t. $v^\top A(k) \leq \beta_v v^\top$, (12)
- $\exists w \in \mathbb{R}^n_{>0}, \exists \beta_w \in (0,1)$ s.t. $A(k)w \leq \beta_w w$. (13)

Proof: We start proving that (12) implies the exponential stability of (11). Define $V(x) = v^\top x(k)$, for $v \in \mathbb{R}^n_{>0}$. Now, compute:

$$V(x(k+1)) = v^\top x(k+1) = v^\top A(k)x(k) \tag{14}$$

$$\stackrel{(12)}{\leq} \beta_v v^\top x(k) = \beta_v V(x(k)), \tag{15}$$

which implies that $V(x(k)) \leq \beta_v^k V(x_0)$ for $k \geq 0$, proving the exponential stability of (11) with guaranteed convergence rate $\beta_v \in (0,1)$.

To see that (13) implies the exponential stability of (11), recalling that $x(k) \geq 0$ at all k, consider the Lyapunov function

$$V(x) = \max_{i=1,\ldots,n} \frac{x_i}{w_i}, \tag{16}$$

where x_i and w_i simply denote the i-th entries of vectors $x(k) \in \mathbb{R}^n_{\geq 0}$ and $w \in \mathbb{R}^n_{>0}$. One has that $x(k) \leq V(x(k))w$, for any $k \geq 0$. It follows that:

$$A(k)x(k) \leq V(x(k))A(k)w, \quad k \geq 0. \tag{17}$$

From (13) one has:

$$A(k)x(k) \leq V(x(k))\beta_w w, \quad k \geq 0, \tag{18}$$

which considered component-wise yields

$$V(x(k+1)) = \max_{i=1,\ldots,n} \frac{[A(k)x(k)]_i}{w_i} \leq \beta_w V(x(k)), \tag{19}$$

from which $V(x(k)) \leq \beta_w^k V(x_0)$ for $k \geq 0$, that implies the exponential stability of (11) with guaranteed convergence rate $\beta_w \in (0,1)$. ■

The next section will illustrate how the preliminary results introduced above can be applied to the stability of PMJLSs.

III. Mean stability of time-varying PMJLSs

We now consider the time evolution of the mean value of $x(k)$ for the MJLS (5) under positivity constraints.

Let us define $q(k) = [q_1^\top(k) \ \cdots q_N^\top(k)]^\top$ with $q_i(k) = E[x(k)\mathbb{1}_{\theta(k)=i}]$. Then, on the lines of [5], it can be shown that $E[x(k)] = \sum_{i=1}^N q_i(k)$ and

$$q_j(k+1) = \sum_{i=1}^N \pi_{ij} A_i(k) q_i(k), \tag{20}$$

that can be compactly written as

$$q(k+1) = (\Pi^\top \otimes I_n)\ \mathrm{diag}_i[A_i(k)]\, q(k), \tag{21}$$

which can be interpreted as a time-varying model with dynamic matrix $Q(k) := (\Pi^\top \otimes I_n)\ \mathrm{diag}_i[A_i(k)]$.

An application of Prop. 2 to this time-varying system describing the mean evolution a PMJLS (5) yields the following results.

Theorem 3: Consider a PMJLS described by (5), with $A_i(k) \geq 0$ at all $k \geq 0$, for all $i \in \underline{N}$. Consider:

$$Q(k) = (\Pi^\top \otimes I_n)\ \mathrm{diag}_i[A_i(k)]. \tag{22}$$

The following statements are equivalent:

a) The PMJLS is exponentially mean stable;
b) The PMJLS is exponentially 1-moment stable.

Moreover, both statements are implied by either one of the following two sufficient conditions:

c) $\exists v \in \mathbb{R}^{nN}_{>0}, \exists \beta_v \in (0,1)$ s.t. $v^\top Q(k) \leq \beta_v v^\top, \ k \geq 0$;
d) $\exists w \in \mathbb{R}^{nN}_{>0}, \exists \beta_w \in (0,1)$ s.t. $Q(k)w \leq \beta_w w, \ k \geq 0$.

Proof: The equivalence among conditions a) and b) crucially relies on the positivity of the system. If a) holds true, then there exists a positive α, and $\beta \in (0,1)$, s.t. $\|E[x(k)]\| \leq \alpha\beta^k\|x_0\|$, for all $k \geq 0$. Then, consider that $x(k) \geq 0$ at all $k \geq 0$, so that $\|E[x(k)]\|_1 = E[\|x(k)\|_1]$. From the equivalence of norms on finite-dimensional spaces, there exists a positive scalar c such that:

$$E[\|x(k)\|] \leq cE[\|x(k)\|_1] = c\|E[x(k)]\|_1 \leq \tilde{\alpha}\beta^k\|x_0\| \tag{23}$$

for some positive $\tilde{\alpha}$, at all $k \geq 0$. Hence, a) implies b).

Conversely, if b) holds true, there exist a positive α, and $\beta \in (0,1)$, such that $E[\|x(k)\|] \leq \alpha\beta^k\|x_0\|$. Then:

$$\|E[x(k)]\| \leq E[\|x(k)\|] \leq \alpha\beta^k\|x_0\|, \quad k \geq 0. \tag{24}$$

Now let us show that c) $\implies$ a), b). Consider that the mean value of the state of (5) is described by $E[x(k)] = \sum_{i=1}^{N} q_i(k)$, as illustrated above. Hence, $E[x(k)] \xrightarrow{k\to\infty} 0$ exponentially if and only if the time-varying system $q(k+1) = Q(k)q(k)$ is exponentially stable. Condition (12) of Prop. 2 applied on this time-varying system yields the implication c) $\implies$ a), with the guaranteed exponential convergence rate $\beta_v \in (0,1)$. In particular, one has that for some positive α it holds:

$$\|q(k)\| \leq \alpha\beta_v^k \|q(0)\|, \quad k \geq 0. \tag{25}$$

Since $E[x(k)] = \sum_{i=1}^{N} q_i(k)$, one can write that:

$$\begin{aligned}\|E[x(k)]\|_1 &= \left\|\sum_{i=1}^{N} q_i(k)\right\|_1 \leq \sum_{i=1}^{N} \|q_i(k)\|_1 = \|q(k)\|_1 \\ &\leq \tilde{\alpha}\beta^k \|q(0)\|_1 = \tilde{\alpha}\beta^k \|E[x(0)]\|_1 = \tilde{\alpha}\beta^k \|x_0\|_1\end{aligned} \tag{26}$$

for some positive $\tilde{\alpha}$.

Finally, the fact that d) $\implies$ a), b) follows with analogous arguments just considering condition (13) of Prop. 2 applied to system $q(k+1) = Q(k)q(k)$. ■

Remark 1: First of all, we stress that the equivalence among mean stability and 1-moment stability only holds due to the positivity assumption on the system. Moreover, we remark that 1-moment stability is also interesting due to the fact that it implies (regardless of positivity) the practically significant almost sure stability, which is difficult to test even in the case of time-invariant discrete modes [3]. Most of literature on MJLSs with time-invariant subsystems focuses on MSS (which is stronger than 1-moment stability) since necessary and sufficient conditions are available, and due to the fact that it implies almost sure stability. Yet, testing MSS involves stability checks on $Nn^2 \times Nn^2$ matrices (see the discussion in Remark 1 of [16]). The weaker property of 1-moment stability studied above can offer a computationally advantageous and practically meaningful alternative for PMJLSs especially for large scale systems, even in the case of time-invariant subsystems (see [16] for the discrete-time case and [17], [18] for the continuous-time one).

The following Corollary proposes an alternative (but equivalent) statement of conditions c) and d) of Thm. 3.

Corollary 4: Consider a PMJLS described by (5), with $A_i(k) \geq 0$ at all $k \geq 0$, for all $i \in \underline{N}$. Either one of the following two conditions is sufficient for the exponential mean stability (or, equivalently, exponential 1-moment stability) of the system:

- $\exists v_i \in \mathbb{R}^n_{>0}$, $i \in \underline{N}$, s.t.

$$\forall i \in \underline{N}, \quad \sum_{j=1}^{N} \pi_{ij} v_j^\top A_i(k) \leq \beta_v v_i^\top. \tag{27}$$

- $\exists w_i \in \mathbb{R}^n_{>0}$, $i \in \underline{N}$, s.t.

$$\forall j \in \underline{N}, \quad \sum_{i=1}^{N} \pi_{ij} A_i(k) w_i \leq \beta_w w_j. \tag{28}$$

Proof: Consider matrix $Q(k)$ in (22):

$$Q(k) = \begin{bmatrix} \pi_{11}A_1(k) & \pi_{21}A_2(k) & \cdots & \pi_{N1}A_N(k) \\ \pi_{12}A_1(k) & \pi_{22}A_2(k) & \cdots & \pi_{N2}A_N(k) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{1N}A_1(k) & \pi_{2N}A_2(k) & \cdots & \pi_{NN}A_N(k) \end{bmatrix}. \tag{29}$$

Now consider condition c) of Thm. 3, one has:

$$\begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}^\top \begin{bmatrix} \pi_{11}A_1(k) & \cdots & \pi_{N1}A_N(k) \\ \vdots & \ddots & \vdots \\ \pi_{1N}A_1(k) & \cdots & \pi_{NN}A_N(k) \end{bmatrix} \leq \beta_v \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}^\top, \tag{30}$$

yielding

$$\begin{aligned} \pi_{11}v_1^\top A_1(k) + \cdots + \pi_{1N}v_N^\top A_1(k) &\leq \beta_v v_1^\top, \\ &\vdots \\ \pi_{N1}v_1^\top A_N(k) + \cdots + \pi_{NN}v_N^\top A_N(k) &\leq \beta_v v_N^\top, \end{aligned} \tag{31}$$

that is the set of conditions (27).

Similarly, considering condition d) of Thm. 3, we have:

$$\begin{bmatrix} \pi_{11}A_1(k) & \cdots & \pi_{N1}A_N(k) \\ \vdots & \ddots & \vdots \\ \pi_{1N}A_1(k) & \cdots & \pi_{NN}A_N(k) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \leq \beta_w \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \tag{32}$$

yielding

$$\begin{aligned} \pi_{11}A_1(k)w_1 + \cdots + \pi_{N1}A_N(k)w_N &\leq \beta_w w_1, \\ &\vdots \\ \pi_{1N}A_1(k)w_1 + \cdots + \pi_{NN}A_N(k)w_N &\leq \beta_w w_N, \end{aligned} \tag{33}$$

that is the set of conditions (28). ■

IV. PMJLS WITH MODE MATRICES IN POLYTOPES

In this section, we consider PMJLS in which the time-varying matrices associated with the N modes (mode matrices) take values on N polytopes. There exists a rich literature on MJLSs with polytopic-valued transition probability matrices (see, e.g., [19] and references therein). Here, we consider the case in which the dynamic matrices $A_i(k)$ take values in polytopes, and the transition probability matrix is time-homogeneous (an extension to the case of time-inhomogeneous, and even polytopic TP matrices is left for future work). For a list of works on the same topic, see [10], [11] and references therein. We remark that the main focus of literature for polytopic and uncertain MJLSs is mean square stability under no positivity constraints, while we here consider mean (and 1-moment) stability for PMJLSs.

As is well known, a polytope in the space of matrices is defined as the convex hull of a finite set of matrices, which are the polytope's vertices. Let $\mathcal{P}_i$ denote the polytope on which the time-varying mode matrix $A_i(k)$ takes values, let r_i be the number of vertices of $\mathcal{P}_i$, and let $A_i^{(h)}$ be the h-th vertex. Thus, the N polytopes $\mathcal{P}_i$ are formally defined as

$$\mathcal{P}_i = \mathrm{conv}(\{A_i^{(1)}, \ldots, A_i^{(r_i)}\}), \quad i = 1, \ldots, N. \tag{34}$$

Clearly, $A_i(k) \in \mathcal{P}_i$ means that at each time instant k there exists a non-negative vector $\eta_i(k) \in \mathbb{R}^{r_i}_{\geq 0}$ such that

$$A_i(k) = \sum_{h=1}^{r_i} A_i^{(h)} \eta_{i,h}(k); \quad \text{with} \quad \sum_{h=1}^{r_i} \eta_{i,h}(k) = 1, \quad (35)$$

where the notation $\eta_{i,h}(k)$ has been used to denote the h-th component of vector $\eta_i(k)$.

A situation in which it is appropriate to consider switching systems with mode matrices taking values in polytopes is the case of uncertain switching systems, that is the case where the mode matrices are not exactly known. In this case, an effective modeling approach is to consider nominal mode matrices $\bar{A}_i$ and bounded, possibly time-varying, uncertainties $\Delta_i(k)$ that take values in a polytope:

$$x(k+1) = \bar{A}_{\theta(k)} x(k) + \Delta_{\theta(k)}(k) x(k), \quad k \geq 0, \ \theta(k) \in \underline{N},$$
$$\text{with } \Delta_i(k) \in \bar{\mathcal{P}}_i^{\Delta} = \text{conv}(\{\Delta_i^{(1)}, \ldots, \Delta_i^{(r_i)}\}). \quad (36)$$

This uncertain PJMLS is equivalent to model (5) by setting

$$A_i(k) = \bar{A}_i + \Delta_i(k), \quad (37)$$

provided that all matrices $\bar{A}_i + \Delta_i(k)$ are non-negative. This is equivalent to assuming that the uncertain time-varying mode matrices $A_i(k)$ belong to the polytopes (34), where the vertices are defined as $A_i^{(h)} = \bar{A}_i + \Delta_i^{(h)}$.

For PMJLS with mode matrices taking values in polytopes, the following Theorem provides a condition of mean and 1-moment stability.

Theorem 5: Consider a PMJLS described by (5), with $A_i(k)$ taking values on polytopes $\mathcal{P}_i$ as in (34), for all $i \in \underline{N}$, where all vertices of all polytopes are non-negative matrices. If the following conditions are verified

$$\begin{aligned} &\exists \beta_v \in (0,1), \ \exists v_i \in \mathbb{R}^n_{>0}, \ i \in \underline{N}, \ \text{s.t.} \\ &\sum_{j=1}^{N} \pi_{ij} v_j^\top A_i^{(h)} \leq \beta_v v_i^\top, \quad \begin{matrix} i = 1, \ldots, N, \\ h = 1, \ldots, r_i, \end{matrix} \end{aligned} \quad (38)$$

then the PMJLS (5) is exponentially mean stable (or, equivalently, exponentially 1-moment stable).

Proof: Recall that $A_i(k) \in \mathcal{P}_i$ means that there exists a $\eta_i(k) \in \mathbb{R}^{r_i}_{\geq 0}$ such that the identities (35) are satisfied. Then, if all conditions (38) are satisfied, for each $i \in \underline{N}$ we can consider the linear combinations of both terms of the r_i inequalities with coefficients $\eta_{i,h}(k)$, for $h = 1, \ldots, r_i$, i.e.

$$\sum_{h=1}^{r_i} \eta_{i,h}(k) \sum_{j=1}^{N} \pi_{ij} v_j^\top A_i^{(h)} \leq \sum_{h=1}^{r_i} \eta_{i,h}(k) \beta_v v_i^\top. \quad (39)$$

Recalling that $\sum_{h=1}^{r_i} \eta_{i,h}(k) = 1$, and distributing the coefficients $\eta_{i,h}(k)$ in the summation, we get

$$\sum_{h=1}^{r_i} \sum_{j=1}^{N} \pi_{ij} v_j^\top A_i^{(h)} \eta_{i,h}(k) \leq \beta_v v_i^\top. \quad (40)$$

From this, rearranging the summations we get, for all $i \in \underline{N}$,

$$\sum_{j=1}^{N} \pi_{ij} v_j^\top \sum_{h=1}^{r_i} A_i^{(h)} \eta_{i,h}(k) \leq \beta_v v_i^\top, \quad (41)$$

Recalling (35), we see that the previous inequalities are equivalent to the inequalities (27), and this, thanks to Cor. 4, proves the exponential mean stability of system (5). ■

Remark 2: According to Theorem 5, the number of vector inequalities to be satisfied to asses the stability of PMJLSs with mode matrices in polytopes is $K = \sum_{i=1}^{N} r_i$, which is also the total number of vertices of the polytopes $\mathcal{P}_i$. Consequently, the number of scalar inequalities is nK. Note that only the inequalities (27) of Cor. 4 have been exploited to provide conditions of 1-moment exponential stability. After a careful investigation, we can say that there is no equally simple way to exploit inequalities (28) for assessing the 1-moment exponential stability for this class of PJMLS.

Clearly, if the unknowns (decision variables) in the K inequalities (38) are the N positive vectors $v_i \in \mathbb{R}^n_{>0}$ and the scalar $\beta_v \in (0,1)$, then the problem is not a standard LP, due to the presence of the products $\beta_v v_j^\top$. Indeed, only if β_v is fixed (not an unknown variable), the problem of finding the N positive vectors v_i that solve the inequalities becomes standard. A deeper discussion on how transform the problem with unknown β_v into a LP so that a standard solver can be used to find the N positive vectors that solve inequalities (38) *for some* $\beta_v \in (0,1)$ is left for a future work due to space constraints. We here remark that this reformulation can be achieved by fairly standard techniques in optimization.

V. Example

Here we demonstrate the paper's results using an illustrative example. Due to space constraints, a single example is presented that focuses on the polytopic case of Sec. IV.

Consider a PMJLS as in (5), with $N = 3$ modes, with the polytopic mode structure described in (34)-(35). The non-negative matrices $A_1(k)$, $A_2(k)$, $A_3(k)$ take value in polytopes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ with vertices given by:

$$\begin{aligned} &A_1^{(1)} = \begin{bmatrix} 0.2 & 0.7 \\ 0.5 & 0.6 \end{bmatrix}, A_1^{(2)} = \begin{bmatrix} 0.3 & 0.9 \\ 0.6 & 0.7 \end{bmatrix}, \\ &A_2^{(1)} = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.2 \end{bmatrix}, A_2^{(2)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}, A_2^{(3)} = \begin{bmatrix} 0 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} \\ &A_3^{(1)} = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0.4 \end{bmatrix}, A_3^{(2)} = \begin{bmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{bmatrix}, \end{aligned} \quad (42)$$

and the transition probability matrix is given by:

$$\Pi = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.5 \\ 0.4 & 0.2 & 0.4 \end{bmatrix}. \quad (43)$$

We remark that $A_1^{(1)}$, $A_1^{(2)}$, and $A_2^{(2)}$ are unstable, i.e. $\rho(A_1^{(1)}), \rho(A_1^{(2)})$, and $\rho(A_2^{(2)})$ are strictly greater than one.

The system in this example is exponentially mean stable (and exponentially 1-moment stable) since it satisfies the conditions of Thm. 5, for example, with $\beta_v = 0.9844$ and

$$v_1 = \begin{bmatrix} 38.4728 \\ 63.7015 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 35.6982 \\ 33.4794 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 22.8460 \\ 16.5126 \end{bmatrix}.$$

The state evolution of the system is depicted in Figs. 1 and 2 for a single realization of $\theta(k)$, and for 1 000 randomly

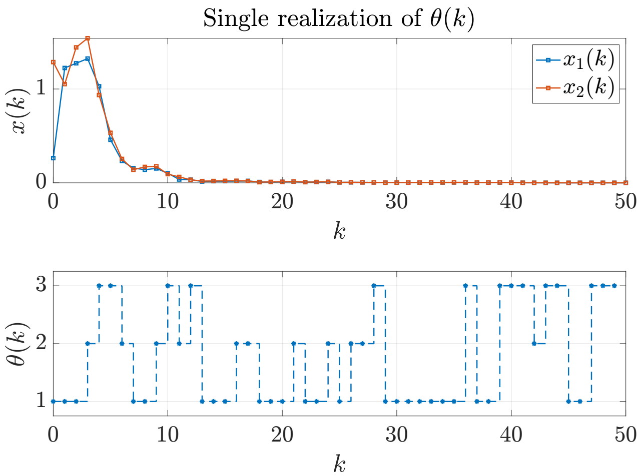

Fig. 1. Example: Time evolution of $x(k)$ (all components) for a single, randomly generated realization of $\theta(k)$.

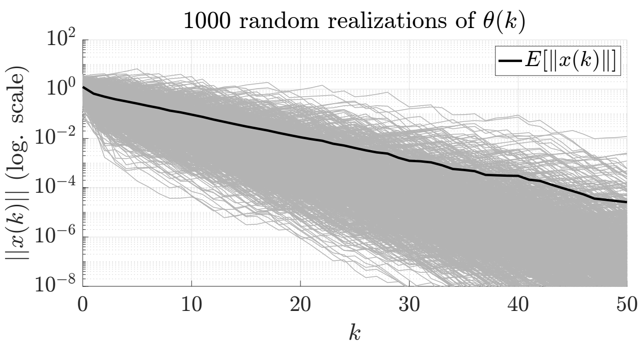

Fig. 2. Example: Time evolution of $\|x(k)\|_2$ over 1000 randomly generated realizations of $\theta(k)$.

generated realizations, respectively. For all such realizations the pairs $x(0), \theta(0)$, and the coefficients $\eta_{i,h}(k)$ of the convex combinations of the vertices of $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are all randomly generated with suitable uniform probability distributions. In particular, $\eta_{i,h}(k)$ are randomly generated to satisfy (35).

The system clearly shows exponential 1-moment convergence, and this also implies exponential almost sure stability.

VI. Discussion and Conclusion

This work established novel sufficient conditions of exponential mean and first-moment stability for positive MJLSs with time-varying subsystems. We adopted an approach based on linear co-positive Lyapunov functions that provides guaranteed bounds on the exponential decay rate. Due to the usage of constant Lyapunov functions on time-varying systems, the conditions are certainly conservative. On the other hand, time-varying Lyapunov conditions would increase the computational burden even more, leading to numerical intractability.

For the special case of uncertain subsystems taking values within polytopes, we have seen how one of the novel stability conditions can be finitely tested. The proposed conditions, even when specialized to subclasses such as polytopic and uncertain PMJLSs, offer a simpler and computationally less demanding approach to deduce almost sure stability (which is implied by 1-moment stability) with respect to the popular analysis based on mean square stability.

Concerning future works, a first idea is to exploit the results proposed on PMJLSs in this paper also for MJLSs with no positivity constraints, on the lines of our previous works [16], [17], [18] which only considered time-invariant discrete modes. Other research ideas pertain to the case of time-varying transition probabilities, and applications. Among these, we mention moving-target defense mechanisms and analysis of control systems under deadline misses.

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